Analyse mathématique/*Mathematical Analysis*

ORTHOGONAL HARMONIC ANALYSIS AND SCALING OF FRACTAL MEASURES
ANALYSE HARMONIQUE ORTHOGONALE DES MESURES FRACTALES AVEC STRUCTURE D'ÉCHELLE

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Abstract

We show that certain iteration systems lead to fractal measures admitting exact orthogonal harmonic analysis.

Résumé

On montre que certains systèmes itératifs conduisent aux mesures fractales qui admettent une analyse harmonique orthogonale exacte.

Version française abrégée

Soit Ω un sous-ensemble mesurable pour la mesure de Lebesgue m sur l'espace Euclidien $\mathbb{R}^d$, $d \geq 1$. Soit $L^2(m_\Omega)$ l'espace de Hilbert des fonctions de carré m_Ω-intégrable par rapport au produit scalaire $\langle f \mid g \rangle := \int \overline{f(x)}\, g(x)\, dm_\Omega(x)$, où m_Ω est la restriction de la mesure de Lebesgue à Ω. Le problème de savoir pour quels sous-ensembles de mesure finie Ω il existe une base orthogonale $\{e_\lambda(x) := \exp(i2\pi\lambda \cdot x) : \lambda \in \Lambda\}$ dans $L^2(m_\Omega)$ a été soulevé par I.E. Segal (1957) et, dans l'article [3], de B. Fuglede, et a été étudié dans [3, 5, 6, 11, 12, 13]. Il est bien connu (cf. [3, 12]) qu'un sous-ensemble ouvert, connexe, et de mesure finie Ω de $\mathbb{R}^d$ admet une base orthogonale si et seulement si la famille des dérivées partielles $-i\frac{\partial}{\partial x_j}$ qui opèrent sur $C_c^\infty(\Omega)$, l'espace des fonctions lisses et à support compact dans Ω, admet par extension une famille d'opérateurs hermitiens H_j, $1 \leq j \leq d$, fortement deux-à-deux commutatifs (dans le sens qu'ils ont des résolutions spectrales commutatives).

Quand $\Omega = [0,1]^d$ est un cube dans $\mathbb{R}^d$, la classe de tous ces opérateurs a été découverte dans [8], avec des résultats particulièrement explicites pour $d \leq 3$. Des domaines Ω admettant de tels opérateurs d'extension mais qui satisfont seulement à une forme plus faible de commutativité ont été étudiés auparavant par J. Friedrich [2] dans le cas $d = 2$.

On étudie une mesure autosimilaire μ avec support contenu dans l'intervalle $[0,1]$ et telle que le sous-espace vectoriel engendré par l'ensemble des fonctions analytiques $\left\{e^{i2\pi nx} : n = 0, 1, 2, \dots\right\}$ soit dense dans $L^2(\mu)$. On identifie selon la dimension fractale de μ les sous-ensembles $P \subset \mathbb{N}_0 = \{0, 1, 2, \dots\}$ tels que les fonctions $\left\{e_n := e^{i2\pi nx} : n \in P\right\}$ constituent une base orthogonale de $L^2(\mu)$. On donne aussi en dimension plus grande une construction affine qui conduit aux mesures autosimilaires μ ayant leurs supports dans $\mathbb{R}^d$. Celle-ci est obtenue à partir d'une matrice expansive d'ordre d et d'un ensemble fini de vecteurs de translation. En plus, pour que l'espace $L^2(\mu)$ correspondant ait une base orthogonale de fonctions exponentielles $e^{i2\pi\lambda\cdot x}$ ayant des vecteurs λ dans $\mathbb{R}^d$ comme ensemble d'indices, il faut que certaines conditions géométriques (qui ont des rapports avec les opérateurs de transfert de Ruelle) sur le système affine soient remplies.

On cite et discute ci-dessous quelques conjectures concernant les mesures qui admettent une analyse harmonique orthogonale exacte.

Soit Ω un sous-ensemble mesurable de mesure finie pour la mesure de Lebesgue sur $\mathbb{R}^d$. S'il existe un ensemble d'indices Λ tel que $\{e_\lambda : \lambda \in \Lambda\}$ est une base orthogonale de $L^2(\Omega)$, alors on dit que Ω est un *ensemble spectral,* Λ est le *spectre,* et (Ω, Λ) est une *paire spectrale.* S'il existe un ensemble T tel que, modulo des ensembles negligeables, la famille $\{\Omega + t : t \in T\}$ est une partition de $\mathbb{R}^d$, alors on dit que Ω est un *pavé* et que T est un *ensemble de pavage.* On a alors la conjecture suivante de Fuglede.

Conjecture 1 (Conjecture de Fuglede [3])**.** *Soit Ω un ensemble de mesure finie et positive. Alors Ω est un ensemble spectral si et seulement si Ω est un pavé.*

Cette conjecture reste ouverte dans les deux sens même pour le cas $d = 1$.

Un ensemble S qui est l'image d'un ensemble de la forme $\mathbb{Z}^d + A$ pour certain ensemble fini A tel que $(A - A) \cap \mathbb{Z}^d = \varnothing$ est dit *périodique*, ou un *treillis avec une base.*

Conjecture 2 (Conjecture sur les ensembles spectraux périodiques [14])**.** *Soit Ω un ensemble de mesure finie et positive. Alors Ω est un ensemble spectral admettant un spectre périodique si et seulement si Ω est un pavé admettant un ensemble de pavage périodique.*

Il a été démontré (cf. [3, 5, 12]) que Ω est un ensemble spectral ayant pour spectre $\mathbb{Z}^d$, si et seulement si, Ω est un pavé avec l'ensemble de pavage $\mathbb{Z}^d$. Les articles [11] et [13] réduisent la conjecture sur les ensembles spectraux périodiques à certaines questions concernant des sous-ensembles finis du treillis $\mathbb{Z}^d$. Pour $d = 1$ des progrès ont été achevés vers la résolution de ces problèmes dans [15]. Ces résultats supportent le point de vue selon lequel certaines classes spécifiques d'ensembles spectraux correspondent à certaines classes d'ensembles de pavage. D'ailleurs ceci est confirmé par quelques résultats dans [8], où nous montrons que tout ensemble spectral périodique du cube $\Omega = [0, 1]^d$ est aussi un ensemble de pavage pour le cube.

1. Introduction

We consider, in this note and in [8] and [9], generalized spectral transforms for a certain Fourier duality in $\mathbb{R}^d$. Our results are motivated by considerations of the transform

$$\xi \longmapsto \int_\Omega e^{-i\xi \cdot x} f(x)\, dx, \quad \xi \in \mathbb{R}^d,\ f \in L^2(\Omega),$$

for a given measurable subset $\Omega \subset \mathbb{R}^d$ of finite Lebesgue measure. Instead we consider pairs of measures (μ, ν) on $\mathbb{R}^d$ such that the following generalized transform,

$$F_\mu f \colon \lambda \longmapsto \int e^{-i2\pi\lambda \cdot x} f(x)\, d\mu(x), \tag{1.1}$$

induces an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$, specifically making precise the following unitarity:

$$\int |f(x)|^2\, d\mu(x) = \int |(F_\mu f)(\lambda)|^2\, d\nu(\lambda).$$

When applied to the case when μ is a measure of compact support with *fractal* Hausdorff dimension, we identify some candidates for pairs (μ, ν), in concrete examples, when duality does hold.

2. Pairs of Measures

2.1. **New Pairs from Old Pairs.** Let μ and ν be Borel measures on $\mathbb{R}^d$. We say that (μ, ν) is a *spectral pair* if the map F_μ from (1.1) above, defined for $f \in L^1 \cap L^2(\mu)$, extends by continuity to an isometric isomorphism mapping $L^2(\mu)$ onto $L^2(\nu)$. It was shown in [12] that if μ is the restriction of Lebesgue measure to a connected set of infinite measure, then the result on extensions of the directional derivatives, described above, remains valid.

2.2. **Which Measures are Possible?** It turns out that the class of measures that can be part of a spectral pair is fairly limited: specifically, we have

Theorem 2.1 (Uncertainty Principle)**.** *Suppose (μ, ν) is a spectral pair, $f \in L^2(\mu)$, $f \neq 0$, and $A, B \subset \mathbb{R}^d$. If $\|f - \chi_A f\|_\mu \leq \varepsilon$ and $\|Ff - \chi_B Ff\|_\mu \leq \delta$, then $(1 - \varepsilon - \delta)^2 \leq \mu(A)\nu(B)$.*

Theorem 2.2 (Local Translation Invariance)**.** *Suppose (μ, ν) is a spectral pair, and $t \in \mathbb{R}^d$. If $\mathcal{O}$ and $\mathcal{O} + t$ are subsets of the support of μ, then $\mu(\mathcal{O}) = \mu(\mathcal{O} + t)$.*

Our work on generalized spectral pairs is motivated by M.N. Kolountzakis and J.C. Lagarias who in [10] discuss related tilings of the real line $\mathbb{R}^1$ by a function.

The following result establishes a direct connection to the spectral pairs mentioned above.

Theorem 2.3. *Suppose (μ, ν) is a spectral pair. If $\mu(\mathbb{R}^d) < \infty$, then ν must be a counting measure with uniformly discrete support.*

3. Fractal Measures

3.1. **Dual Iteration Systems.** Consider a triplet (R, B, L) such that R is an expansive $d \times d$ matrix with real entries, and B and L are subsets of $\mathbb{R}^d$ such that $N := \#B = \#L$,

$$R^n b \cdot l \in \mathbb{Z}, \text{ for any } n \in \mathbb{N},\ b \in B,\ l \in L, \tag{3.1}$$

$$H_{B,L} := N^{-1/2} \left(e^{i2\pi b \cdot l} \right)_{b \in B,\, l \in L} \text{ is a unitary } N \times N \text{ matrix.} \tag{3.2}$$

We introduce two dynamical systems, $\sigma_b(x) := R^{-1}x + b$ and $\tau_l(x) := R^* x + l$, and the corresponding "attractors", $X_\sigma := \left\{ \sum_{k=0}^{\infty} R^{-k} b_k : b_k \in B \right\}$ and

$$\mathcal{L} = X_\tau := \left\{ \sum_{k=0}^{n} R^{*k} l_k : n \in \mathbb{N},\ l_k \in L \right\}. \tag{3.3}$$

The set X_ρ is then the support of the unique probability measure which solves the equation

$$\mu = N^{-1} \sum_{b \in B} \mu \circ \sigma_b^{-1}. \tag{3.4}$$

We show that, under certain geometric assumptions, the exponentials $E(\mathcal{L}) = \{e_\lambda : \lambda \in \mathcal{L}\}$ form an orthogonal basis for $L^2(\mu)$. It follows from the assumptions on (R, B, L) that $E(\mathcal{L})$ is orthogonal; so the question is whether or not these exponentials span all of $L^2(\mu)$. If we set

$$\chi_B(t) := N^{-1} \sum_{b \in B} e_b(t), \tag{3.5}$$

then the expansiveness property of R and (3.4) imply an explicit product formula for the Fourier transform of μ,

$$\widehat{\mu}(t) := \int \overline{e_t(x)}\, d\mu(x) = \prod_{k=0}^{\infty} \chi_B(R^{*\,-k} t), \tag{3.6}$$

the convergence being uniform on bounded subsets of $\mathbb{R}^d$. We introduce the function

$$Q(t) := \sum_{\lambda \in \mathcal{L}} |\widehat{\mu}(t - \lambda)|^2, \quad t \in \mathbb{R}^d, \tag{3.7}$$

and the Ruelle operator C given by

$$(Cq)(t) := \sum_{l \in L} |\chi_B(t - l)|^2 \, q(\rho_l(t)), \tag{3.8}$$

where $\rho_l(x) := R^{*\,-1}(x - l)$. Both Q and the constant function $\mathbf{1}$ are eigenfunctions for the Ruelle operator C with eigenvalue 1, and the issue becomes one of multiplicity. The attractor

$$X_\rho := \left\{ \sum_{k=0}^{\infty} -R^{*\,-k} l_k : l_k \in L \right\}, \tag{3.9}$$

corresponding to the system $\{\rho_l\}$, will also be used below.

3.2. **Orthogonal Bases.** Let $H_2(\mathcal{L})$ denote the subspace of $L^2(\mu)$ spanned by the orthonormal set $\{e_\lambda : \lambda \in \mathcal{L}\}$. Any e_t, $t \in \mathbb{C}^d$ is in $L^2(\mu)$ so $H_2(\mathcal{L})$ is a subspace of $L^2(\mu)$. We will show that $H_2(\mathcal{L}) = L^2(\mu)$ for specific systems (R, B, L) satisfying (3.1)–(3.2).

Let Y denote the convex hull of the attractor X_ρ given by (3.9), and let $\|q\|_\infty := \sup_{y \in Y} |q(y)|$. We then introduce the following Lipschitz norm:

$$\|q\|_{Y,\infty} := \| |\nabla q|_2 \|_\infty . \tag{3.10}$$

We show that, if the operator norm of C, acting on a suitable set of smooth functions, is less than one, then μ has the basis property.

Theorem 3.1. *Let (R, B, L) be a system in $\mathbb{R}^d$ satisfying (3.1)–(3.2), $0 \in L$. Let C be the operator given by (3.8), let Y denote the convex hull of the attractor X_ρ given by (3.9), and let $\|q\|_{Y,\infty}$ be given by (3.10). Supposing that L spans $\mathbb{R}^d$, if there exists $\gamma < 1$ such that $\|Cq\|_{Y,\infty} \leq \gamma \|q\|_{Y,\infty}$ for all q in a set of C^1-functions containing $\mathbf{1} - Q$, then $H_2(\mathcal{L}) = L^2(\mu)$.*

The following result on Lipschitz estimates allows us to compute an explicit and numerical operator norm bound γ for C in terms of the given data (R, B, L):

Theorem 3.2. *Let (R, B, L) be a system in $\mathbb{R}^d$ satisfying (3.1)–(3.2), $0 \in L$. Let C be the operator given by (3.8), and let Y denote the convex hull of the attractor X_ρ given by (3.9). Let $\|q\|_{Y,\infty}$ be given by (3.10), and*

$$\beta := 2\pi \operatorname{diam}(B) \max_{\substack{b,b' \in B \\ l \in L}} \|\sin(2\pi(b - b')(\,\cdot\, - l))\|_\infty .$$

Then we have the estimate

$$\|Cq\|_{Y,\infty} \leq \left[(N-1)^2 N^{-1} \beta \left\|R^{-1}\right\|_{\mathrm{op}} \max_{l \in L} |l|_2 + \left\|R^{-1}\right\|_{\mathrm{hs}} \right] \|q\|_{Y,\infty} ,$$

for all C^1-functions q such that $q(0) = 0$. Here $\|T\|_{\mathrm{op}}$ is the operator norm, and $\|T\|_{\mathrm{hs}}$ is the Hilbert–Schmidt norm for a $d \times d$ matrix T.

Corollary 3.3. *Let (R, B, L) satisfy (3.1)–(3.2), and for $r \in \mathbb{N}$ let*

$$\mathcal{L}_r := \left\{ \sum_{k=0}^{n} (rR^*)^k \, l_k : n \in \mathbb{N}, l_k \in L \right\}$$

and let μ_r be the probability measure solving $\mu_r = N^{-1} \sum_{b \in B} \mu_r \circ \sigma_{r,b}^{-1}$, where $\sigma_{r,b}(x) := (rR)^{-1} x + b$ (i.e., a scaled version of (3.4)). If L spans $\mathbb{R}^d$ and $0 \in L$, then, provided r is sufficiently large, it follows that $\{e_\lambda : \lambda \in \mathcal{L}_r\}$ is an orthonormal basis for $L^2(\mu_r)$.

R.S. Strichartz obtained an asymptotic and quite different harmonic analysis for the class of measures considered in this paper: it was based instead on a continuous transform (see [16] for a survey of Strichartz's work on self-similarity in harmonic analysis).

4. Applications

4.1. **Fractal Hardy Spaces.** One way to construct systems (R, B, L) satisfying (3.1)–(3.2) is to pick R, B and L so that

$$R \in M_d(\mathbb{Z}), \quad RB \subset \mathbb{Z}^d, \quad L \subset \mathbb{Z}^d. \tag{4.1}$$

In fact (4.1) implies (3.1) since $R^n b \cdot l = Rb \cdot R^{*(n-1)} l$ for $n = 1, 2, 3, \ldots$. The only condition that is hard to satisfy is (3.2). This last condition is notoriously difficult: for example, it is not known which matrices with entries in the unit circle satisfy (3.2) for $N = 7$; see, e.g., [1, 4] for some progress in the study of (3.2). The condition (4.1) is closely related to a condition use in the study of certain multi-dimensional wavelets. Some results for systems (R, B, L) satisfying (4.1) and (3.2) were established in [7].

If R has non-negative integer entries, we will often end up with $\{e_\lambda : \lambda \in \mathcal{L}\}$ being an orthonormal basis for $L^2(\mu)$, and each element in $\mathcal{L}$ only having non-negative coordinates. This is an interesting situation because the basis property leads to the expansion $f = \sum_{\lambda \in \mathcal{L}} \langle e_\lambda \mid f \rangle_\mu e_\lambda$, so setting $z_j := e^{i 2\pi x_j}$ we see that $f(x) = \sum_{\lambda \in \mathcal{L}} \langle e_\lambda \mid f \rangle_\mu z^\lambda$ for $f \in L^2(\mu)$, where $z^\lambda := \prod_{k=1}^d z_k^{\lambda_k}$. It follows that $f(x)$, $x \in X_\sigma$, is the a.e. boundary value of a function analytic in the polydisc $\{z \in \mathbb{C}^d : |z_j| < 1\}$. Hence our construction shows that many fractal L^2-spaces are Hardy spaces. This is in sharp contrast to the Lebesgue spaces, for example, if $m_{[0,1]}$ is Lebesgue measure restricted to the unit interval $[0, 1]$.

4.2. **Examples.** Using Theorem 3.2, Theorem 3.1, and equation (3.6), one can prove the following result.

Theorem 4.1. *Suppose $d = 1$, $N = 2$, $B = \{0, a\}$, with $a \in \mathbb{R} \setminus \{0\}$, R is an integer with $|R| \geq 2$, and μ is given by (3.4). If R is odd, then $L^2(\mu)$ does not have a basis of exponentials for any $a \in \mathbb{R} \setminus \{0\}$. If R is even and $|R| \geq 4$, then $L^2(\mu)$ has a basis of exponentials for all $a \in \mathbb{R} \setminus \{0\}$.*

Using convolution and Theorem 4.1 one can verify the following example.

Example 4.2. *Let μ_0 be the probability measure solving (3.4) when $R = 4$ and $B = \{0, 1/2\}$. Let $L = \{0, 1\}$ and let $\mathcal{L}$ be given by (3.3). Set $\Omega := [0, 1] + \mathcal{L}$, and define two measures μ and ν: $\mu(\Delta) := m(\Delta \cap \Omega)$, $\nu(\Delta) := \sum_{k=0}^\infty \mu_0(\Delta + k)$. Then (μ, ν) is a spectral pair, and Ω is a tile with tiling set $-2\mathcal{L}$.*

This is an example of a spectral set of infinite measure whose spectrum is not periodic, and it takes us full circle, connecting back to the two extension problems discussed in Conjectures 1 and 2 above.

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